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Permutations of separable preference orders☆

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Abstract

The notion of separability is important in economics, operations research, and political science, where it has recently been studied within the context of referendum elections. In a referendum election on n questions, a voter's preferences may be represented by a linear order on the 2^n possible election outcomes. The symmetric group of degree 2^n, S_{2^n}, acts in a natural way on the set of all such linear orders. A permutation $\sigma \in S_{2^n}$ is said to *preserve separability* if for each separable order $\succ$, $\sigma(\succ)$ is also separable. Here, we show that the set of separability-preserving permutations is a subgroup of S_{2^n} and, for 4 or more questions, is isomorphic to the Klein 4-group. Our results indicate that separable preferences are rare and highly sensitive to small changes. The techniques we use have applications to the problem of enumerating separable preference orders and to other broader combinatorial questions.

Keywords: Separable preferences; Permutations; Symmetric group; Inversion numbers

1. Introduction

Actors in decision-making processes are often required to voice simultaneously their preferences on several possibly related issues. A classic example is the referendum election, in which voters cast ballots simultaneously on multiple questions or proposals. According to Lacy and Niou [8], "the resurrection of direct democracy through referendums is one of the clear trends of democratic politics." But referendums are not without flaw. In fact, Brams et al. [3] identify what they call the "paradox of multiple elections," a voting scenario in which no individual voter casts a ballot that agrees with the outcome of the election on each issue. Lacy and Niou [8] further demonstrate that the winning outcome in such an election can in fact be the last choice of every voter. They argue that this paradoxical behavior occurs because "referendums as currently practiced force people to separate their votes on issues that may be linked in their minds."

The phenomenon to which Lacy and Niou refer is known as the separability problem [2]. What they and others have observed is that voter preferences often contain interdependencies that cannot be expressed through the standard method of voting in a referendum. For example, a voter's preferences on one bond proposal may depend on the outcome of another, especially if both proposals draw funds from the same tax base. By requiring a simultaneous vote on both issues, referendum elections provide no adequate means of expressing this interdependence. Thus, no ballot can accurately represent the voter's true preferences, a fact that seems to imply many of the aforementioned difficulties.

☆ Much of this work also appeared in the author's doctoral dissertation [5].
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There has been relatively little research done on ways to solve the separability problem. Alternative voting methods, such as election sequencing, have been proposed as potential solutions, but preliminary investigations of such methods have been a cause for skepticism among some researchers. For instance, Kilgour and Bradley [7] observe that "election sequencing can produce results that are socially worse than the simultaneous election." They suggest that "one way out of the difficulty may be to frame questions so as to avoid preference nonseparability."

Is this suggestion realistic? Can we reasonably expect to avoid the possibility of interdependence within voter preferences? Our investigations here suggest not. In particular, we show that separable preferences (that is, preferences free from interdependence) are rare and highly sensitive to small changes. We do so by investigating the structure of separability from within a combinatorial and group theoretic framework. Specifically, we show that the likelihood of separability approaches zero as the number of questions in an election increases, and that the group of permutations that preserve separability contains only four elements (and is in fact isomorphic to the Klein 4-group) for elections with four or more questions. The techniques we use have potential applications to the problem of enumerating separable preference orders and to other broader counting problems.

2. Definitions, notation, and background results

We begin by adopting a slight variation of the preference model used by Bradley et al. [1], which is based on that of Yu [9]. As noted above, we assume the context of a referendum election[1] on a finite set Q of n questions, where $n \geqslant 2$. By an *outcome*, we mean an n-tuple $(x_1, x_2, \ldots, x_n)$, where $x_i \in \{0, 1\}$ for each i.[2] We denote by X_Q the set of all such outcomes, noting that $|X_Q| = 2^n$. If S is a nonempty, proper subset of Q and $x \in X_Q$, then we let x_S denote the components of x corresponding to the questions in S. Furthermore, if $\mathscr{P} = \{S_1, S_2, \ldots, S_n\}$ is a partition of Q, then we distinguish between the components of x by writing $x = (x_{S_1}, x_{S_2}, \ldots, x_{S_n})$, reordering the questions if necessary.[3] Specifically, we denote by $-S$ the complement of S in Q and write $x = (x_S, x_{-S})$.

We use a strict linear order $\succ$ on X_Q to represent a voter's preferences over the possible outcomes of an election on Q, sometimes calling $\succ$ a *binary preference order* on Q, or simply a *preference order*. The sequence of outcomes (x_k) satisfying $x_1 \succ x_2 \succ \cdots \succ x_{2^n}$ is called the *order sequence* corresponding to $\succ$. In this context, x_1 is said to be the *leader* of $\succ$ since $x_1 \succ x$ for all $x \in X_Q$ with $x \neq x_1$. Finally, the $2^n \times n$ matrix whose kth row is x_k is called the *binary preference matrix* corresponding to $\succ$. Binary preference matrices provide a convenient way to represent a voter's preferences, as we will see shortly in Example 4.

2.1. Separability

Intuitively, a voter's preferences on a subset S of Q are said to be *separable* if they do not depend on the outcome of the election on questions outside of S. Formally:

Definition 1. Let $S \subset Q$. Then S is said to be $\succ$-*separable*, or *separable with respect to* $\succ$, if whenever two elements $x_S, y_S \in X_S$ have the property that

$$(x_S, u_{-S}) \succ (y_S, u_{-S})$$

for some $u_{-S} \in X_{-S}$, then

$$(x_S, v_{-S}) \succ (y_S, v_{-S})$$

[1] Formally, we can define a referendum election to be a function $\mathscr{E}$ whose domain is the set of all multisets of linear orders on X_Q and whose range is X_Q. In this way, $\mathscr{E}$ assigns to each set of voter preferences a unique outcome. The manner in which this outcome in determined is irrelevant to our investigations.

[2] For ease of notation, we often omit parentheses and commas. For example, we write 1101 instead of (1, 1, 0, 1).

[3] We allow the parts of $\mathscr{P}$ to be empty, in which case we take the notation $x = (x_{S_1}, x_{S_2}, \ldots, x_{S_n})$ to mean $x = (x_{S_{i_1}}, x_{S_{i_2}}, \ldots, x_{S_{i_m}})$, where $\{i_k : 1 \leqslant k \leqslant m\}$ is the set of indices corresponding to the nonempty parts of $\mathscr{P}$.

for all $v_{-S} \in X_{-S}$. A question $q \in Q$ is said to be $\succ$-separable if $\{q\}$ is $\succ$-separable. The preference order $\succ$ is said to be *separable* if each nonempty $S \subset Q$ is $\succ$-separable. In this case, we call $\succ$ a *separable preference order*.

In addition to Definition 1, we adopt the convention that both Q and $\emptyset$ are always $\succ$-separable. In most cases, we will only be interested in subsets that are both nonempty and proper. If a set S is $\succ$-separable, then we sometimes say that $\succ$ is *separable on S*.

Note that if a subset S of Q is $\succ$-separable, then $\succ$ induces a linear order $\succ_S$ on X_S defined by

$$x_S \succ_S y_S \iff (x_S, u_{-S}) \succ (y_S, u_{-S}) \quad \text{for all } u_{-S} \in X_{-S}.$$

This induced order satisfies the following properties:

Proposition 2. *Let $\succ$ be a linear order on X_Q and let $S \subset Q$ be $\succ$-separable. If $T \subset S$ is $\succ$-separable, then T is $\succ_S$-separable and $(\succ_S)_T = \succ_T$.*

Proof. Let $T \subset S$ and suppose that $(x_T, u_{S-T}) \succ_S (y_T, u_{S-T})$ for some $x_T, y_T \in X_T$ and some $u_{S-T} \in X_{S-T}$. Let $v_{S-T} \in X_{S-T}$ be given. Then by the definition of $\succ_S$, $(x_T, u_{S-T}, w_{-S}) \succ (y_T, u_{S-T}, w_{-S})$ for all $w_{-S} \in X_{-S}$. But since T is $\succ$-separable, it follows that $(x_T, v_{S-T}, w_{-S}) \succ (y_T, v_{S-T}, w_{-S})$ for all $w_{-S} \in X_{-S}$. Thus $(x_T, v_{S-T}) \succ_S (y_T, v_{S-T})$ and, consequently, T is $\succ_S$-separable. To show $(\succ_S)_T = \succ_T$, we observe that

$$\begin{aligned} x_T (\succ_S)_T y_T &\iff (x_T, u_{S-T}) \succ_S (y_T, u_{S-T}) \quad \text{for all } u_{S-T} \in X_{S-T} \\ &\iff (x_T, u_{S-T}, v_{-S}) \succ (y_T, u_{S-T}, v_{-S}) \quad \text{for all } u_{S-T} \in X_{S-T}, v_{-S} \in X_{-S} \\ &\iff (x_T, w_{-T}) \succ (y_T, w_{-T}) \quad \text{for all } w_{-T} \in X_{-T} \\ &\iff x_T \succ_T y_T. \qquad \square \end{aligned}$$

Corollary 3. *Let $\succ$ be a separable order on X_Q and let $S \subset Q$. Then $\succ_S$ is a separable order on X_S.*

Proof. This is an immediate consequence of Proposition 2. $\square$

Example 4. Suppose $|Q| = n = 3$ and consider the linear order $\succ$ on X_Q specified by the following binary preference matrix:

$$R = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}.$$

In this example, both $\{3\}$ and $\{1, 2\}$ are separable with induced orders $1 \succ 0$ and $10 \succ 01 \succ 00 \succ 11$, respectively. In contrast, $\{1\}$ is nonseparable since $101 \succ 001$ (inducing an order of $1 \succ 0$ on $\{1\}$) but $011 \succ 111$ (inducing an order of $0 \succ 1$ on $\{1\}$). Similarly, $\{2\}$ is nonseparable since $101 \succ 111$ but $011 \succ 001$. Thus, we see that it is possible for a set to be separable even if none of its proper subsets is.

2.2. All separable preference matrices for $n = 2, 3, 4$

A preference order $\succ$ on X_Q is said to be *normalized* if its leader is $(1, 1, 1, \ldots, 1)$ and if

$$(1, 0, 0, \ldots, 0) \succ (0, 1, 0, \ldots, 0) \succ (0, 0, 1, \ldots, 0) \succ \cdots \succ (0, 0, 0, \ldots, 1).$$

In this case, we say also that the binary preference matrix corresponding to $\succ$ is *normalized*. Note that every binary preference matrix can be obtained from some normalized matrix by simply permuting and/or taking bitwise complements of the columns (that is, replacing ones with zeros and vice versa). Thus, if we wish to determine all possible

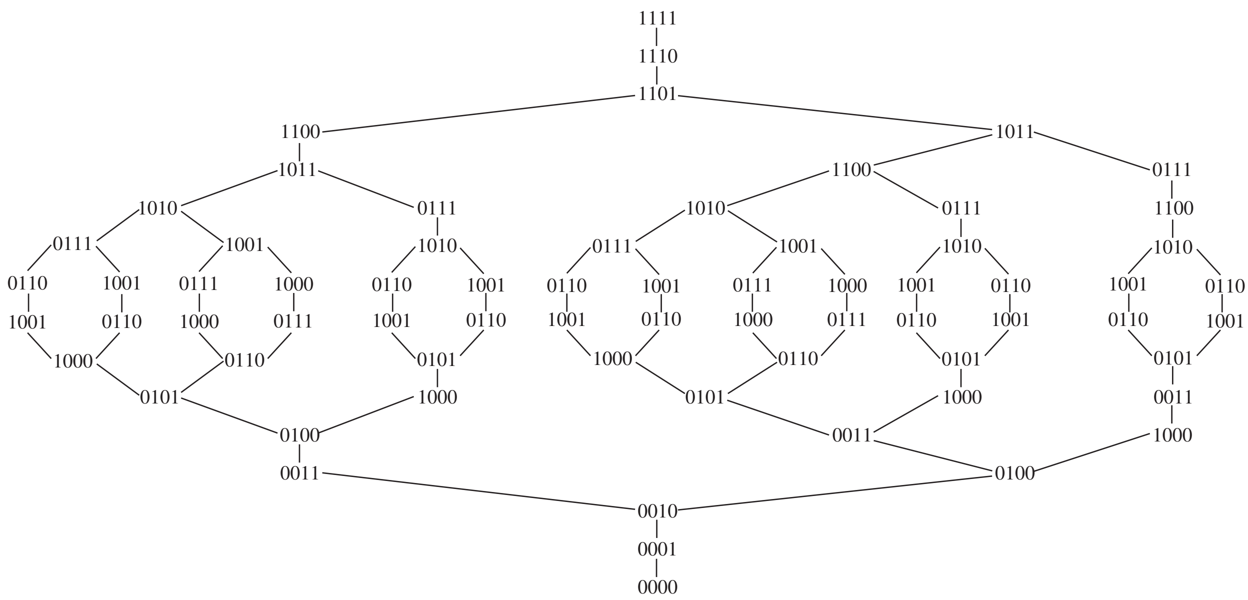

Fig. 1. Normalized separable preference matrices on 4 questions.

separable preference orders for Q, it suffices to find all of the normalized separable preference orders. Bradley et al. do exactly this in [1]. They show that, for $n = 2$, the matrix

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

is the only normalized separable preference matrix.

For $n = 3$, there are exactly two normalized, separable preference matrices,

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

For $n = 4$, there are 14 normalized separable preference matrices. The digraph summarizes these—the nodes of any directed path from 1111 to 0000 will give a separable preference matrix in Fig. 1.

2.3. Lexicographic sums

In this section, we develop a useful tool for constructing separable preference orders.

Definition 5. Let Q_1 and Q_2 be disjoint question sets and let $\succ_1$, $\succ_2$ be linear orders on X_{Q_1} and X_{Q_2}, respectively. The *lexicographic sum* of $\succ_1$ and $\succ_2$ is the linear order $\succ_1 \oplus \succ_2$ on $X_{Q_1 \cup Q_2}$ defined by

$$(x_{Q_1}, x_{Q_2})(\succ_1 \oplus \succ_2)(y_{Q_1}, y_{Q_2}) \iff x_{Q_1} \succ_1 y_{Q_1} \quad \text{or} \quad (x_{Q_1} = y_{Q_1} \quad \text{and} \quad x_{Q_2} \succ_2 y_{Q_2}).$$

Proposition 6. *If $\succ_1$ and $\succ_2$ are separable orders on X_{Q_1} and X_{Q_2}, respectively, then $\succ_1 \oplus \succ_2$ is a separable order on $X_{Q_1 \cup Q_2}$.*

Proof. Let $\succ\, = \succ_1 \oplus \succ_2$ and let $S \subset Q_1 \cup Q_2$ be given. Define $S_1 = Q_1 \cap S$ and $S_2 = Q_2 \cap S$. Then, $S = S_1 \cup S_2$ and $-S = (Q_1 - S_1) \cup (Q_2 - S_2)$. Now suppose that

$$(x_{S_1}, x_{S_2}, u_{Q_1-S_1}, u_{Q_2-S_2}) \succ (y_{S_1}, y_{S_2}, u_{Q_1-S_1}, u_{Q_2-S_2})$$

for some $x, y \in X_S$ and some $u \in X_{-S}$. Let $v \in X_{-S}$. We must show that

$$(x_{S_1}, x_{S_2}, v_{Q_1-S_1}, v_{Q_2-S_2}) \succ (y_{S_1}, y_{S_2}, v_{Q_1-S_1}, v_{Q_2-S_2}). \qquad (\dagger)$$

By Definition 5, one of the following must occur:

(i) $(x_{S_1}, u_{Q_1-S_1}) \succ_1 (y_{S_1}, u_{Q_1-S_1})$, in which case the separability of $\succ_1$ implies that $(x_{S_1}, v_{Q_1-S_1}) \succ_1 (y_{S_1}, v_{Q_1-S_1})$, which then implies (†); or
(ii) $(x_{S_1}, u_{Q_1-S_1}) = (y_{S_1}, u_{Q_1-S_1})$ and $(x_{S_2}, u_{Q_2-S_2}) \succ_2 (y_{S_2}, u_{Q_2-S_2})$, in which case $x_{S_1} = y_{S_1}$, and so $(x_{S_1}, v_{(Q_1-S_1)}) = (y_{S_1}, v_{(Q_1-S_1)})$. The separability of $\succ_2$ then implies that $(x_{S_2}, v_{Q_2-S_2}) \succ_2 (y_{S_2}, v_{Q_2-S_2})$, which implies (†).

Since each case implies (†), we have shown that S is $\succ$-separable. Since our choice of S was arbitrary, it follows that $\succ$ is separable. □

It can be shown easily that the lexicographic sum is an associative binary operation. As such, we generally omit parenthesis when taking multiple lexicographic sums. For instance, we write $\succ_1 \oplus \succ_2 \oplus \succ_3$ instead of $(\succ_1 \oplus \succ_2) \oplus \succ_3$. Note that the lexicographic sum is not a commutative operation. Indeed, the order in which the components are summed plays an essential role in Definition 5.

Proposition 7. *Let Q_1 and Q_2 be disjoint question sets, where $|Q_1| = p$ and $|Q_2| = q$. Let $\succ_1$ be a linear order on X_{Q_1} with order sequence (x_k) and let $\succ_2$ be linear order on X_{Q_2} with order sequence (y_k). Let (z_k) be the order sequence corresponding to $\succ_1 \oplus \succ_2$. Then, for all integers i and j with $0 \leqslant i \leqslant 2^p - 1$ and $1 \leqslant j \leqslant 2^q$,*

$$z_{i(2^q)+j} = (x_{i+1}, y_j).$$

Proof. Let $\succ\, = \succ_1 \oplus \succ_2$. Observe that, by the definition of the lexicographic sum,

$$\begin{aligned}(x_k, y_l) \succ (x_{i+1}, y_j) \iff & \; k < i + 1 \\ & \text{or} \quad k = i + 1 \quad \text{and} \quad l < j.\end{aligned}$$

Since there are $i(2^q)$ elements of the first type and $j - 1$ elements of the second type, exactly $i(2^q) + j - 1$ elements precede (x_{i+1}, y_j) in the order sequence for $\succ$, as desired. □

We conclude our investigation of lexicographic sums by examining a related class of orders that will be of significant use to us in the proofs of subsequent results.

Definition 8. Let $\succ$ be a linear order on X_Q. If, for each $q \in Q$, there exists a linear order $\succ_q$ on X_q such that

$$\succ\, = \succ_{\sigma(1)} \oplus \succ_{\sigma(2)} \oplus \cdots \oplus \succ_{\sigma(n)}$$

for some $\sigma \in S_n$, then $\succ$ is said to be a *lexicographic order* on Q. The permutation σ is called the *importance permutation* of the order.

Note that a lexicographic order is uniquely determined by its leader and its importance permutation. Thus:

Proposition 9. *There are exactly $2^n \cdot n!$ distinct lexicographic orders on Q.*

Also notice that, since every linear order on a single question is vacuously separable, Proposition 6 implies that every lexicographic order is separable. This result has been proved in prior work (see [6], for example), but follows here as a result of a more general theory of lexicographic sums.

The normalized lexicographic order (with leader $(1, 1, \ldots, 1)$ and importance permutation equal to the identity permutation) will be of special use to us. We often call this order the *standard lexicographic order* on Q, denoted by $\succ_{\text{lex}}$. The standard lexicographic order has the useful property that the kth row of the binary preference matrix corresponding to $\succ_{\text{lex}}$ is, loosely speaking, the binary expansion of $2^n - k$ (with leading zeros possibly added). More precisely:

Proposition 10. *Let $n \geqslant 2$ and let $(a_{i,j})$ be the binary preference matrix corresponding to $\succ_{\text{lex}}$. Then, for each positive integer $i \leqslant 2^n$,*

$$\sum_{j=1}^{n} a_{i,j} \cdot 2^{n-j} = 2^n - i.$$

Proof. We proceed by induction. For $n = 2$, the result can be easily verified by examining the four rows of the standard lexicographic order on two questions. Now suppose that it is true for some integer $n \geqslant 2$. For each positive integer k, let $\succ_k$ denote the standard lexicographic order on k questions, so that $\succ_{n+1} = \succ_n \oplus \succ_1$. Let $(a_{i,j})$, $(b_{i,j})$ be the binary preference matrices corresponding to $\succ_n$ and $\succ_{n+1}$, respectively. Proposition 7 implies that

$$b_{2i-1,j} = b_{2(i-1)+1,j} = \begin{cases} a_{i,j} & \text{if } 1 \leqslant j \leqslant n, \\ 1 & \text{if } j = n+1. \end{cases}$$

Similarly,

$$b_{2i,j} = b_{2(i-1)+2,j} = \begin{cases} a_{i,j} & \text{if } 1 \leqslant j \leqslant n, \\ 0 & \text{if } j = n+1. \end{cases}$$

Thus, by the induction hypothesis,

$$\sum_{j=1}^{n+1} b_{2i-1,j} \cdot 2^{n+1-j} = 1 + \sum_{j=1}^{n} a_{i,j} \cdot 2^{n+1-j} = 1 + 2(2^n - i) = 2^{n+1} - (2i - 1)$$

and

$$\sum_{j=1}^{n+1} b_{2i,j} \cdot 2^{n+1-j} = \sum_{j=1}^{n} a_{i,j} \cdot 2^{n+1-j} = 2(2^n - i) = 2^{n+1} - 2i,$$

as desired. □

2.4. Symmetric orders

For an outcome $x \in X_Q$, let $\overline{x}$ denote the bitwise complement of x (so that if, for example, $x = (1, 0, 1)$, then $\overline{x} = (0, 1, 0)$). Definition 11 and Propositions 12 and 13 are originally due to Bradley et al. [1].

Definition 11. Let $\succ$ be a linear order on X_Q. Then $\succ$ is said to satisfy the *mirror property* if, for all $x, y \in X_Q$, $x \succcurlyeq y \implies \overline{y} \succcurlyeq \overline{x}$.

Proposition 12. *A linear order $\succ$ on X_Q satisfies the mirror property if and only if its order sequence (x_k) is such that $x_{2^n-i+1} = \overline{x}_i$ for each positive integer $i \leqslant 2^n$.*

Proposition 13. *If $\succ$ is a separable preference order on X_Q, then $\succ$ satisfies the mirror property.*

If $\succ$ is a linear order on X_Q and $\succ$ satisfies the mirror property, then we say that $\succ$ is a *symmetric order*.

Proposition 14. *There are exactly* $2^{2^{n-1}} \cdot 2^{n-1}!$ *symmetric orders on* Q.

Proof. Let (x_k) be the order sequence corresponding to $\succ$. Note that every symmetric order on Q can be uniquely determined by specifying $x_1, x_2, \ldots, x_{2^{n-1}}$ so that $x_i \neq \overline{x_j}$ for all $i, j \leqslant 2^{n-1}$ (the bottom half of the order sequence is then determined by the mirror property). Furthermore, all such choices of $x_1, x_2, \ldots, x_{2^{n-1}}$ correspond to symmetric orders. Now x_1 may be chosen in 2^n ways, whereas x_2 may be chosen in $2^n - 2$ ways (x_2 cannot be equal to x_1 or $\overline{x_1}$), x_3 may be chosen in $2^n - 4$ ways, and so on. Thus, the number of symmetric orders on Q is equal to

$$2^n \cdot (2^n - 2) \cdot (2^n - 4) \cdots 4 \cdot 2 = 2^{2^{n-1}} \cdot 2^{n-1}! \qquad \square$$

No closed formula currently exists for determining the number of separable preference orders on a given question set. Proposition 14, however, does provide some insight into the rarity of separable preferences. Let $P(m)$ denote the probability that a randomly selected binary preference order on a finite question set of cardinality m is separable. Then:

Corollary 15. $P(m) \to 0$ *as* $m \to \infty$.

Proof. For each positive integer m, let $s(m)$ denote the number of separable preference orders on a question set of cardinality m. Note that the total number of preference orders on such a question set is $2^m!$. By Propositions 13 and 14, $s(m) \leqslant 2^{2^{m-1}} \cdot 2^{m-1}!$, and so

$$\begin{aligned} P(m) &= \frac{s(m)}{2^m!} \\ &\leqslant \frac{2^{2^{m-1}} \cdot 2^{m-1}!}{2^m!} \\ &= \frac{2^m \cdot (2^m - 2) \cdot (2^m - 4) \cdots 4 \cdot 2}{2^m \cdot (2^m - 1) \cdot (2^m - 2) \cdots 2 \cdot 1} \\ &= \frac{1}{(2^m - 1) \cdot (2^m - 3) \cdot (2^m - 5) \cdots 3 \cdot 1}, \end{aligned}$$

which approaches 0 (very quickly!) as $m \to \infty$. $\square$

Proposition 14 also implies that, when $|Q| = n = 2$, there are exactly eight symmetric orders on X_Q. Notice, however, that since each separable preference matrix can be obtained from a normalized matrix by permuting and/or taking bitwise complements of the columns, and since there is exactly one normalized separable preference matrix for $n = 2$, it follows that there are exactly eight *separable* orders on X_Q as well. Since each separable order is also symmetric, we have the following:

Proposition 16. *For* $n = 2$, *separability and symmetry are equivalent*.

3. Symmetry-preserving permutations

We have now developed the machinery we need to commence upon our main goal of investigating the behavior of the symmetric group as it acts on collections of binary preference matrices. We begin by formally defining a useful group action, and we then briefly study the effect of this action on the property of symmetry. Because every separable order is also symmetric (by Proposition 13), the results in this section will constitute our first steps toward a characterization of the group of separability-preserving permutations.

Assume $n \geqslant 2$ and let $\mathcal{O}_n$ denote the set of all linear orders on X_Q. We define an action of the symmetric group S_{2^n} on the set $\mathcal{O}_n$ as follows: Let $\succ \in \mathcal{O}_n$ be given and let (x_k) be the order sequence corresponding to $\succ$. For each $\sigma \in S_{2^n}$, define $\sigma(\succ)$ to be the linear order whose order sequence (y_k) is given by $y_{\sigma(k)} = x_k$, or more conveniently, $y_k = x_{\sigma^{-1}(k)}$. Observe that $\sigma(\succ)$ is well-defined since σ is a permutation and is thus bijective. The map $(\sigma, \succ) \mapsto \sigma(\succ)$, which we call the *canonical action*, is a group action of S_{2^n} on $\mathcal{O}_n$. Note that the canonical action is both faithful and transitive and that the stabilizer of any order $\succ \in \mathcal{O}_n$ is trivial.

Now, recall that if $\succ$ is a symmetric order on X_Q with corresponding order sequence (x_k), then $\overline{x}_k = x_{2^n-k+1}$ for each $1 \leqslant k \leqslant 2^n$. For convenience and to be consistent with this notation, we define $\overline{k} = 2^n - k + 1$, so that a preference order $\succ$ is symmetric if and only if $\overline{x}_k = x_{\overline{k}}$ for each k.

Definition 17. A permutation $\sigma \in S_{2^n}$ is said to *preserve symmetry* if for each symmetric order $\succ$ on X_Q, $\sigma(\succ)$ is also symmetric.

We denote by $\overline{S}_{2^n}$ the set of all symmetry-preserving permutations in S_{2^n}. Similarly, we denote by $\overline{\mathcal{O}}_n$ the set of all symmetric orders on X_Q.

Proposition 18. *The set $\overline{S}_{2^n}$ of symmetry-preserving permutations is a subgroup of S_{2^n}.*

Proof. The identity permutation clearly preserves symmetry, and so $1 \in \overline{S}_{2^n}$. Since $\overline{S}_{2^n}$ is a finite subset of S_{2^n}, it now suffices to show that $\overline{S}_{2^n}$ is closed under function composition. Thus, let $\sigma, \tau \in S_{2^n}$ and let $\succ$ be a symmetric order on X_Q. Since τ preserves symmetry, $\tau(\succ)$ is symmetric. But since σ preserves symmetry, $(\sigma\tau)(\succ) = \sigma(\tau(\succ))$ is then symmetric, as desired. □

It can be shown that $\overline{S}_{2^n}$ is isomorphic to the group of symmetries of a 2^{n-1}-dimensional hypercube (see Hodge [5] for details). For the remainder of this section, we focus on a few simple properties of the elements of $\overline{S}_{2^n}$.

Proposition 19. *A permutation $\sigma \in S_{2^n}$ preserves symmetry if and only if*

$$\sigma(r) = s \implies \sigma(\overline{r}) = \overline{s}$$

for all r, s.

Proof. Let $\succ$ be any symmetric order on X_Q, and let (x_k) and (y_k) be the order sequences corresponding to $\succ$ and $\sigma(\succ)$, respectively.

($\Rightarrow$) Suppose that σ preserves symmetry and that $\sigma(r) = s$ for some r and s. By the definition of the canonical action, $y_s = y_{\sigma(r)} = x_r$. But since σ preserves symmetry, both $\succ$ and $\sigma(\succ)$ are symmetric. Thus, $y_{\overline{s}} = \overline{y_s} = \overline{x_r} = x_{\overline{r}}$, which implies that $\sigma(\overline{r}) = \overline{s}$, as desired.

($\Leftarrow$) For the converse, suppose that $\sigma(r) = s \implies \sigma(\overline{r}) = \overline{s}$ for all r and s. Then for each k, $\overline{\sigma^{-1}(k)} = \sigma^{-1}(\overline{k})$ and so

$$\overline{y}_k = \overline{x}_{\sigma^{-1}(k)} = x_{\overline{\sigma^{-1}(k)}} = x_{\sigma^{-1}(\overline{k})} = y_{\overline{k}},$$

which implies that $\sigma(\succ)$ is symmetric. Since our choice of $\succ$ was arbitrary, it follows that σ preserves symmetry. □

Proposition 20. *Let $\succ$ be symmetric. Then $\sigma(\succ)$ is symmetric if and only if σ preserves symmetry.*

Proof. The reverse implication is immediate. Thus, assume that $\sigma(\succ)$ is symmetric and assume, to the contrary, that σ does not preserve symmetry. Let (x_k) and (y_k) be the order sequences corresponding to $\succ$ and $\sigma(\succ)$, respectively. By Proposition 19, there exist r and s such that $\sigma(r) = s$ and $\sigma(\overline{r}) \neq \overline{s}$. But then, by the definition of the canonical action, $y_s = x_r$ and $y_{\overline{s}} \neq x_{\overline{r}}$. Since both $\succ$ and $\sigma(\succ)$ are symmetric, this then implies that $\overline{y_s} = y_{\overline{s}} \neq x_{\overline{r}} = \overline{x_r} = \overline{y_s}$, a contradiction. □

Proposition 21. *If σ preserves symmetry and $\sigma(\succ)$ is symmetric, then $\succ$ is symmetric.*

Proof. Suppose that σ preserves symmetry and that $\sigma(\succ)$ is symmetric. By Proposition 18, σ^{-1} preserves symmetry. But then it must be the case that $\succ = \sigma^{-1}(\sigma(\succ))$ is symmetric, as desired. □

Note that the contrapositive of Proposition 21 says that symmetry-preserving permutations must also preserve asymmetry. Specifically, if σ preserves symmetry and $\succ$ is asymmetric, then $\sigma(\succ)$ must also be asymmetric.

4. Separability-preserving permutations

We now consider the set of permutations that preserve the property of separability.

Definition 22. A permutation $\sigma \in S_{2^n}$ is said to *preserve separability* if for each separable order $\succ$ on X_Q, $\sigma(\succ)$ is also separable.

We denote by $S^*_{2^n}$ the set of separability-preserving permutations. Similarly, we denote by $\mathcal{O}^*_n$ the collection of all separable preference orders on X_Q. Our ultimate goal is to classify the structure of $S^*_{2^n}$ (up to isomorphism) for all n. In this section, we establish some basic properties of $S^*_{2^n}$ that will help us to achieve this goal.

Proposition 23. *The set $S^*_{2^n}$ of separability-preserving permutations is a subgroup of $\overline{S}_{2^n}$.*

Proof. Let $\sigma \in S^*_{2^n}$ and choose some $\succ \in \mathcal{O}^*_n$. Since σ preserves separability and $\succ$ is separable, it follows that $\sigma(\succ)$ is separable. But then Proposition 13 implies that both $\succ$ and $\sigma(\succ)$ are symmetric. It then follows from Proposition 20 that σ preserves symmetry. Thus, $\sigma \in \overline{S}_{2^n}$ and so we have shown that $S^*_{2^n} \subseteq \overline{S}_{2^n}$. The proof that $S^*_{2^n}$ is a subgroup is analogous to the proof of Proposition 18. $\square$

Proposition 24. *If σ preserves separability and $\sigma(\succ)$ is separable, then $\succ$ is separable.*

Proof. The result is immediate, since $\succ = \sigma^{-1}(\sigma(\succ))$ and, by Proposition 23, σ^{-1} preserves separability. $\square$

We note here (analogous to our remarks following Proposition 21) that Proposition 23 implies that a separability-preserving permutation also preserves nonseparability. Unfortunately, not all of the nice properties of $\overline{S}_{2^n}$ carry over to their analogs in $S^*_{2^n}$. For example, consider the action of the permutation $\sigma = (2, 4, 7, 5)$ on $\succ_{\text{lex}}$:

$$\begin{pmatrix} 1&1&1\\1&1&0\\1&0&1\\1&0&0\\0&1&1\\0&1&0\\0&0&1\\0&0&0 \end{pmatrix} \xrightarrow{\sigma} \begin{pmatrix} 1&1&1\\0&1&1\\1&0&1\\1&1&0\\0&0&1\\0&1&0\\1&0&0\\0&0&0 \end{pmatrix} \xrightarrow{\sigma} \begin{pmatrix} 1&1&1\\0&0&1\\1&0&1\\0&1&1\\1&0&0\\0&1&0\\1&1&0\\0&0&0 \end{pmatrix}.$$

Notice that both $\succ_{\text{lex}}$ and $\sigma(\succ_{\text{lex}})$ are separable, and yet $\sigma(\sigma(\succ_{\text{lex}})) = \sigma^2(\succ_{\text{lex}})$ is nonseparable. It follows that σ does not preserve separability.

Then we see that, while the property of preserving symmetry is global in the sense that if σ preserves the symmetry of one symmetric order, then σ preserves the symmetry of all symmetric orders, the property of preserving separability is more localized. Specifically, it is possible for a permutation to preserve the separability of one separable order while failing to preserve the separability of another. To formalize this local property, we make the following definition:

Definition 25. Let $\succ$ be a separable order on X_Q. A permutation $\sigma \in S_{2^n}$ is said to (*locally*) *preserve the separability of* $\succ$ if $\sigma(\succ)$ is separable.

A consequence of this definition is the following:

Proposition 26. *A permutation $\sigma \in S_{2^n}$ preserves separability if and only if σ preserves the separability of each $\succ \in \mathcal{O}^*_n$.*

For a separable order $\succ$, we denote by $S^{\succ}_{2^n}$ the set of permutations which preserve the separability of $\succ$. In the special case where $\succ = \succ_{\text{lex}}$, we write $S^{\text{lex}}_{2^n}$ instead.

Notice that if σ preserves the separability of some separable preference order $\succ$, then both $\succ$ and $\sigma(\succ)$ are symmetric (by Proposition 13), from which it follows by Proposition 20 that σ preserves symmetry. Thus, $S_{2^n}^{\succ} \subseteq \overline{S}_{2^n}$ for each $\succ \in \mathcal{O}_n^*$.

Note also that, by the previous example, $S_{2^n}^{\text{lex}}$ is not a subgroup of $\overline{S}_{2^n}$ (since σ preserves the separability of $\succ_{\text{lex}}$ and σ^2 does not). A similar result holds for all $\succ \in \mathcal{O}_n^*$, though we must delay the proof of this result until we have built up some stronger machinery (see Proposition 65).

We conclude this section with a combinatorial result.

Proposition 27. *Let* $\succ \in \mathcal{O}_n^*$. *Then* $|S_{2^n}^{\succ}| = |\mathcal{O}_n^*|$.

Proof. We prove the claim by exhibiting a bijection between $S_{2^n}^{\succ}$ and $\mathcal{O}_n^*$. Let $\varphi : S_{2^n}^{\succ} \to \mathcal{O}_n^*$ be the map defined by $\varphi(\sigma) = \sigma(\succ)$. Since $\succ$ is separable and σ preserves the separability of $\succ$, φ is well-defined. To see that φ is injective, recall that

$$\sigma_1(\succ) = \sigma_2(\succ) \iff \sigma_2^{-1}\sigma_1 \text{ stabilizes } \succ \iff \sigma_2^{-1}\sigma_1 = 1 \iff \sigma_1 = \sigma_2.$$

To see that φ is surjective, note that, for any $\succ' \in \mathcal{O}_n^*$, there exists $\sigma \in S_{2^n}$ such that $\sigma(\succ) = \succ'$ (by the transitivity of the canonical action). But since $\succ$ and $\succ'$ are both separable, it follows that $\sigma \in S_{2^n}^{\succ}$. Thus φ is surjective, as desired. $\square$

Corollary 28. *For all* $\succ_1, \succ_2 \in \mathcal{O}_n^*$, $|S_{2^n}^{\succ_1}| = |S_{2^n}^{\succ_2}|$.

Note that, by Proposition 27, the problem of enumerating separable preference orders is equivalent to that of enumerating the permutations which preserve the separability of some particular order $\succ \in \mathcal{O}_n^*$. Note that the choice of $\succ$ here is irrelevant, though some orders, such as $\succ_{\text{lex}}$, may be more convenient to work with than others.

5. The structure of $S_{2^n}^*$

We now turn our attention to the task of determining the structure of the group of separability-preserving permutations, $S_{2^n}^*$. Our main result is the following, of which part (iii) will be of most interest to us.

Theorem 29. *Let* V_4 *denote the Klein* 4-*group and let* D_4 *denote the group of symmetries of the square* (*the dihedral group of degree* 4).

(i) *For* $n = 2$, $S_{2^n}^* = S_4^* \cong D_4$.
(ii) *For* $n = 3$, $S_{2^n}^* = S_8^* \cong V_4 \times S_3$.
(iii) *For* $n \geqslant 4$, $S_{2^n}^* \cong V_4$.

Our argument is, for the most part, combinatorial in nature. The cases for $n \leqslant 4$ are verified by brute force using the preference orders provided in Section 2.2. An induction argument then completes the proof. Along the way, we actually prove the stronger claim that $\sigma \in S_{2^n}^*$ if and only if its *inversion number* $\text{inv}(\sigma)$ (defined formally below) belongs to the set $\left\{0, 1, \binom{2^n}{2} - 1, \binom{2^n}{2}\right\}$. We then prove that this set of inversion numbers corresponds uniquely to a set of permutations isomorphic to V_4.

The glue that holds the induction together is the claim that if $S_{2^n}^*$ contains a permutation σ for which $1 < \text{inv}(\sigma) < \binom{2^n}{2} - 1$, then $S_{2^{n-1}}^*$ contains a permutation σ' for which $1 < \text{inv}(\sigma') < \binom{2^{n-1}}{2} - 1$. The proof of this fact is lengthy but not overly complex.

We begin by developing some tools that will allow us to translate results about $S_{2^n}^*$ into results about $S_{2^{n-1}}^*$. Let $\succ \in \mathcal{O}_n^*$ and let (x_k) be the order sequence corresponding to $\succ$. Then, we define a map $p_i \colon \mathcal{O}_n^* \to \mathcal{O}_{n-1}^*$, called the ith *projection map* on $\mathcal{O}_n^*$, by $p_i(\succ) = \succ_{Q-\{q_i\}}$. This map is well-defined by Corollary 3.

Lemma 30. *Let* $\succ \in \mathcal{O}_{n-1}^*$ *and let* $\succ_1 \in \mathcal{O}_1$ *(so either* $1 \succ_1 0$ *or* $0 \succ_1 1$*). Then*

(i) $p_1(\succ_1 \oplus \succ) = \succ$,
(ii) $p_n(\succ \oplus \succ_1) = \succ$.

Proof. This follows directly from Definition 5.

For a fixed $\succ \in \mathcal{O}_n^*$, the projection maps defined above induce maps $s_i : S_{2^n}^* \to S_{2^{n-1}}^*$ in the following manner: For $\sigma \in S_{2^n}^*$, let $\sigma' \in S_{2^{n-1}}$ be the unique permutation for which $\sigma'(p_i(\succ)) = p_i(\sigma(\succ))$. Since $\succ$ is separable and σ preserves separability, σ' is well-defined. Now define s_i to be the map which sends σ to σ'.[4]

As noted above, we investigate the separability-preserving behavior of a particular permutation σ by considering the elements of the ordered set $(X_Q, \succ)$ inverted by σ. This property of inversion is defined formally below. The subsequent results establish an elementary theory of inversions, as they relate to our study of separable preference orders.

Definition 31. Let $\sigma \in S_m$ and let a and b be integers such that $1 \leqslant a < b \leqslant m$. We say that σ *inverts* the pair $\langle a, b \rangle$ if $\sigma(a) > \sigma(b)$. We denote by $\mathrm{inv}(\sigma)$ the number of distinct pairs inverted by σ.[5]

Note that, for $\sigma \in S_m$, $0 \leqslant \mathrm{inv}(\sigma) \leqslant \binom{2^m}{2}$. Furthermore, $\mathrm{inv}(\sigma) = 0$ if and only if $\sigma = 1$ and $\mathrm{inv}(\sigma) = \binom{2^m}{2}$ if and only if $\sigma = \tau_r$, where τ_r is the *reflection permutation*, defined by $\tau_r(a) = m - a + 1 = \overline{a}$ for all a.

Proposition 32. *Let* $\succ \in \mathcal{O}_n^*$ *have order sequence* (x_k), *let* $\sigma \in S_{2^n}^*$, *and let* $i \leqslant n$ *be given. Let* (y_k) *be the order sequence corresponding to* $p_i(\succ)$ *and let* $a, b \leqslant 2^n$ *be such that* $x_a = (y_c, j)$ *and* $x_b = (y_d, j)$ *for some* $c, d \leqslant 2^{n-1}$ *and some* $j \in \{0, 1\}$. *Then* σ *inverts* $\langle a, b \rangle$ *if and only if* $s_i(\sigma)$ *inverts* $\langle c, d \rangle$.

Proof. Let $x_a = (y_c, j)$ and $x_b = (y_d, j)$ be as above. Without loss of generality, assume that $x_a \succ x_b$, so that $a < b$. Let $\succ_i = p_i(\succ)$, let $\succ' = \sigma(\succ)$, and let $\succ_i' = p_i(\sigma(\succ)) = p_i(\succ')$. It follows from the definition of $p_i(\succ)$ that $y_c \succ_i y_d$. Now observe that

$$\begin{aligned} \sigma \text{ inverts } \langle a, b \rangle &\iff \sigma(a) > \sigma(b) \\ &\iff (y_d, j) = x_b \succ' x_a = (y_c, j) \\ &\iff y_d \succ_i' y_c. \end{aligned}$$

But since $s_i(\sigma)$ is the unique permutation for which

$$s_i(\sigma)(\succ_i) = s_i(\sigma)(p_i(\succ)) = p_i(\sigma(\succ)) = p_i(\succ') = \succ_i',$$

it follows that σ inverts $\langle a, b \rangle$ if and only if $y_d \succ_i' y_c$, if and only if $s_i(\sigma)$ inverts $\langle c, d \rangle$. □

Lemma 33. *The permutation* $\sigma\tau$ *inverts* $\langle a, b \rangle$ *if and only if one of the following conditions holds*:

(i) τ *inverts* $\langle a, b \rangle$ *and* σ *does not invert* $\langle \tau(b), \tau(a) \rangle$; *or*
(ii) τ *does not invert* $\langle a, b \rangle$ *and* σ *inverts* $\langle \tau(a), \tau(b) \rangle$.

Proof. If either (i) or (ii) holds, then $\sigma(\tau(a)) > \sigma(\tau(b))$, as desired. Now suppose conversely that $\sigma\tau$ inverts $\langle a, b \rangle$. Then $\sigma(\tau(a)) > \sigma(\tau(b))$. If τ inverts $\langle a, b \rangle$, then $\tau(b) < \tau(a)$, which implies that σ does not invert $\langle \tau(b), \tau(a) \rangle$.

[4] Note that s_i depends on our choice of $\succ$. This notational ambiguity will cause no real difficulties, as the choice of $\succ$ will be made clear by the context in which s_i appears.

[5] In this definition, our pairs are unordered, in the sense that we do not distinguish between $\langle a, b \rangle$ and $\langle b, a \rangle$. Our convention will be to list the smaller of a and b first whenever their relative sizes are known.

On the other hand, if τ does not invert $\langle a, b \rangle$, then $\tau(a) < \tau(b)$, which implies that σ inverts $\langle \tau(a), \tau(b) \rangle$. Since τ must either invert or not invert the pair $\langle a, b \rangle$, it follows that either (i) or (ii) must occur. $\square$

Lemma 34. *Let $\sigma \in S_{2^n}$. Then* $\mathrm{inv}(\tau_r \sigma) = \binom{2^n}{2} - \mathrm{inv}(\sigma)$.

Proof. Since τ_r inverts all possible pairs, Lemma 33 implies that $\tau_r\sigma$ inverts $\langle a, b \rangle$ if and only if σ does not invert $\langle a, b \rangle$. The result then follows from the fact that there are $\binom{2^n}{2}$ possible pairs. $\square$

Lemma 35. *If σ inverts $\langle a, b \rangle$, then σ^{-1} inverts $\langle \sigma(b), \sigma(a) \rangle$.*

Proof. Suppose σ inverts $\langle a, b \rangle$. Then $\sigma(b) < \sigma(a)$ and $\sigma^{-1}(\sigma(b)) = b > a = \sigma^{-1}(\sigma(a))$, which implies that σ^{-1} inverts $\langle \sigma(b), \sigma(a) \rangle$. $\square$

The next proposition implies that a permutation is uniquely determined by the set of pairs that it inverts. This set of pairs is sometimes referred to as the *inversion table* of the permutation.

Proposition 36. *If $\sigma \neq \tau$, then there exists a pair $\langle a, b \rangle$ such that $\langle a, b \rangle$ is inverted by exactly one of σ or τ.*

Proof. Suppose $\sigma \neq \tau$. Then $\sigma\tau^{-1} \neq 1$. By our comments following Definition 31, this implies that $\sigma\tau^{-1}$ inverts some pair $\langle a, b \rangle$. Thus, by Lemma 33, it must be the case that either

(i) τ^{-1} inverts $\langle a, b \rangle$ and σ does not invert $\langle \tau^{-1}(b), \tau^{-1}(a) \rangle$; or
(ii) τ^{-1} does not invert $\langle a, b \rangle$ and σ inverts $\langle \tau^{-1}(a), \tau^{-1}(b) \rangle$.

If (i) occurs, then Lemma 35 implies that τ inverts $\langle \tau^{-1}(b), \tau^{-1}(a) \rangle$ (and σ does not). If (ii) occurs, the same lemma implies that τ does not invert $\langle \tau^{-1}(a), \tau^{-1}(b) \rangle$ (and σ does). In either case, the pair $\langle \tau^{-1}(a), \tau^{-1}(b) \rangle$ is inverted by exactly one of σ or τ. $\square$

Proposition 37. *Let $\succ_1 \in \mathcal{O}^*_{n-1}$ and let $\succ_2 \in \mathcal{O}^*_1$. Then, for any $\sigma \in S^*_{2^n}$:*

(i) *The permutation $s_1(\sigma)$ induced by $p_1(\succ_2 \oplus \succ_1)$ does not depend on the choice of $\succ_1$. Furthermore, $s_1(\sigma) \in S^*_{2^{n-1}}$.*
(ii) *The permutation $s_n(\sigma)$ induced by $p_n(\succ_1 \oplus \succ_2)$ does not depend on the choice of $\succ_1$. Furthermore, $s_n(\sigma) \in S^*_{2^{n-1}}$.*

Proof. We prove (i) and leave the analogous proof of (ii) to the reader. Assume, without loss of generality, that $1 \succ_2 0$ and let (x_k), (y_k) be the order sequences corresponding to $\succ_1$ and $\succ_2 \oplus \succ_1$, respectively. Let $\sigma \in S^*_{2^n}$ be given and let $\sigma' = s_1(\sigma) \in S_{2^{n-1}}$. Then σ' is the unique permutation for which $\sigma'(p_1(\succ_2 \oplus \succ_1)) = p_1(\sigma(\succ_2 \oplus \succ_1))$. But by Lemma 30, $p_1(\succ_2 \oplus \succ_1) = \succ_1$ and so σ' is the unique permutation for which $\sigma'(\succ_1) = p_1(\sigma(\succ_2 \oplus \succ_1))$. Thus, for any pair $\langle a, b \rangle$, we have

$$\begin{aligned}
\sigma' \text{ inverts } \langle a, b \rangle &\iff x_b \sigma'(\succ_1) x_a \\
&\iff x_b p_1(\sigma(\succ_2 \oplus \succ_1)) x_a \\
&\iff (1, x_b)\sigma(\succ_2 \oplus \succ_1)(1, x_a) \quad \text{and} \quad (0, x_b)\sigma(\succ_2 \oplus \succ_1)(0, x_a) \\
&\iff y_b \sigma(\succ_2 \oplus \succ_1) y_a \quad \text{and} \quad y_{b+2^{n-1}} \sigma(\succ_2 \oplus \succ_1) y_{a+2^{n-1}} \\
&\qquad \text{(by Proposition 7)} \\
&\iff \sigma \text{ inverts both } \langle a, b \rangle \quad \text{and} \quad \langle a + 2^{n-1}, b + 2^{n-1} \rangle.
\end{aligned}$$

Thus, the inversion table for σ' is completely determined by the inversion table for σ, which clearly does not depend on the choice of $\succ_1$. By Proposition 36, σ' is uniquely determined by its inversion table. Consequently, $\sigma' = s_1(\sigma)$ is uniquely determined by σ alone and hence does not depend on the choice of $\succ_1$. $\square$

To see that $\sigma' = s_1(\sigma) \in S^*_{2^{n-1}}$, observe that, for any $\succ_1 \in \mathcal{O}^*_{n-1}$, $\sigma(\succ_2 \oplus \succ_1)$ is separable and thus, by Corollary 3, $p_1(\sigma(\succ_2 \oplus \succ_1))$ is separable. But then, by Lemma 30 and the definition of $s_1(\sigma)$, we have

$$\sigma'(\succ_1) = \sigma'(p_1(\succ_2 \oplus \succ_1)) = p_1(\sigma(\succ_2 \oplus \succ_1)),$$

which we just observed to be separable. Since $\succ_1$ was chosen arbitrarily, it follows that $\sigma' = s_1(\sigma) \in S^*_{2^{n-1}}$.

Lemma 38. *If σ inverts $\langle a, b \rangle$ and $a < c < b$, then σ inverts either $\langle a, c \rangle$ or $\langle c, b \rangle$.*

Proof. Suppose, to the contrary, that σ inverts neither $\langle a, c \rangle$ nor $\langle c, b \rangle$. Then $\sigma(a) < \sigma(c) < \sigma(b)$, a contradiction to the assumption that σ inverts $\langle a, b \rangle$. $\square$

Lemma 39. *If σ inverts $\langle a, b \rangle$ and $\langle b, c \rangle$, then σ inverts $\langle a, c \rangle$.*

Proof. If σ inverts $\langle a, b \rangle$ and σ inverts $\langle b, c \rangle$, then $\sigma(a) > \sigma(b) > \sigma(c)$, as desired. $\square$

Lemma 40. *Let $\sigma \in \overline{S}_{2^n}$. If σ inverts $\langle a, b \rangle$, then σ inverts $\langle \overline{b}, \overline{a} \rangle$.*

Proof. Suppose σ inverts $\langle a, b \rangle$. Then $\sigma(a) > \sigma(b)$, which implies that

$$\sigma(\overline{b}) = \overline{\sigma(b)} > \overline{\sigma(a)} = \sigma(\overline{a}),$$

as desired. $\square$

Proposition 41. *Let $\sigma \in \overline{S}_{2^n}$. If $\mathrm{inv}(\sigma) = 1$, then $\sigma = (2^{n-1}, 2^{n-1} + 1)$.*

Proof. Suppose $\mathrm{inv}(\sigma) = 1$ and let $\langle a, b \rangle$ be the pair inverted by σ. By Lemma 38, $b = a + 1$, for otherwise there would exist c with $a < c < b$ and hence another inversion. By Lemma 40, σ also inverts $\langle \overline{b}, \overline{a} \rangle = \langle \overline{a+1}, \overline{a} \rangle$, which implies that $a = \overline{a+1} = 2^n - a$. Thus, $a = 2^{n-1}$ and the pair inverted by σ is exactly $\langle 2^{n-1}, 2^{n-1} + 1 \rangle$. Since the permutation $(2^{n-1}, 2^{n-1} + 1)$ inverts this pair and no others, Proposition 36 implies that $\sigma = (2^{n-1}, 2^{n-1} + 1)$. $\square$

We call the permutation $(2^{n-1}, 2^{n-1} + 1)$ the *central transposition*, denoted by τ_{c}.

Lemma 42. *If $\mathrm{inv}(\sigma) = \binom{2^n}{2} - 1$, then $\sigma = \tau_{\mathrm{r}}\tau_{\mathrm{c}}$.*

Proof. If $\mathrm{inv}(\sigma) = \binom{2^n}{2} - 1$, then $\mathrm{inv}(\tau_{\mathrm{r}}\sigma) = 1$ (by Lemma 34). Thus, $\tau_{\mathrm{r}}\sigma = \tau_{\mathrm{c}}$, which implies that $\sigma = \tau_{\mathrm{r}}\tau_{\mathrm{c}}$. $\square$

Definition 43 and Proposition 44 establish a key property of separability-preserving permutations. The corollaries that follow are the first real fruits of our labor in this section.

Definition 43. Let $\succ \in \mathcal{O}_n$ and let (x_k) be the order sequence corresponding to $\succ$. The 4-tuple $\langle a, b, c, d \rangle$, where $a < b$ and $c < d$, is said to be a *complete set* of indices (with respect to $\succ$) if for some $S \subset Q$, there exist $y, z \in X_S$ and $u, v \in X_{-S}$ such that

$$x_a = (y, u), \quad x_c = (y, v),$$
$$x_b = (z, u), \quad x_d = (z, v).$$

Proposition 44. *Let $\sigma \in S_{2^n}$ and let $\succ \in \mathcal{O}^*_n$. Then σ preserves the separability of $\succ$ if and only if, for each complete set $\langle a, b, c, d \rangle$, σ inverts either both or neither of the pairs $\langle a, b \rangle$, $\langle c, d \rangle$.*

Proof. ($\Rightarrow$) We prove the contrapositive. Thus, suppose that there exists a complete set $\langle a, b, c, d \rangle$ such that σ inverts exactly one of $\langle a, b \rangle$ and $\langle c, d \rangle$. Without loss of generality, assume that $\langle a, b \rangle$ is the inverted pair. Let $S \subset Q$ and x_a, x_b, x_c, x_d be as in Definition 43 and let $\succ' = \sigma(\succ)$. Then, $x_b \succ' x_a$ (since $x_a \succ x_b$) and $x_c \succ' x_d$.

This, however, implies that S is not $\succ'$-separable, since $z \succ' y$ given a choice of u on X_{-S} but $y \succ' z$ given a choice of v on X_{-S}. Thus, $\succ' = \sigma(\succ)$ is nonseparable and so σ does not preserve the separability of $\succ$.

($\Leftarrow$) For the converse, suppose that σ does not preserve the separability of $\succ$; that is, suppose that $\succ' = \sigma(\succ)$ is not separable. Then for some $S \subset Q$, there exist $y, z \in X_S$ and $u, v \in X_{-S}$ such that $(z, u) \succ' (y, u)$ and $(y, v) \succ' (z, v)$. But since $\succ$ is separable, either $(y, u) \succ (z, u)$ and $(y, v) \succ (z, v)$ or $(z, u) \succ (y, u)$ and $(z, v) \succ (y, v)$. Without loss of generality, assume the former. Let (x_k) be the order sequence corresponding to $\succ$ and let a, b, c, d be the indices such that

$$\begin{aligned} x_a &= (y, u), \quad x_c = (y, v), \\ x_b &= (z, u), \quad x_d = (z, v). \end{aligned}$$

Then $\langle a, b, c, d \rangle$ is a complete set. Furthermore, σ inverts $\langle a, b \rangle$ (since $x_b \succ' x_a$) but does not invert $\langle c, d \rangle$ (since $x_c \succ' x_d$). $\square$

Corollary 45. *Let $\sigma \in S_{2^n}$ and let $\succ_1, \succ_2 \in \mathcal{O}_n^*$. If $\succ_1$ and $\succ_2$ have the same complete sets, then σ preserves the separability of $\succ_1$ if and only if σ preserves the separability of $\succ_2$.*

Corollary 46. *A permutation σ belongs to $S_{2^n}^*$ if and only if σ preserves the separability of each normalized preference order in $\mathcal{O}_n^*$.*

Proof. The forward implication is immediate. For the converse, suppose that σ preserves the separability of each normalized order in $\mathcal{O}_n^*$. For each $\succ \in \mathcal{O}_n^*$, there exists a normalized preference order $\succ' \in \mathcal{O}_n^*$ such that $\succ$ can be obtained from $\succ'$ by reordering the question and replacing a subset of the components of each outcome with their bitwise complements. This process preserves the complete sets of $\succ'$ and so it follows that $\succ$ and $\succ'$ have the same complete sets. Since σ preserves the separability of $\succ'$, Corollary 45 implies that σ preserves the separability of $\succ$. Since $\succ$ was chosen arbitrarily, it follows that $\sigma \in S_{2^n}^*$. $\square$

Corollary 47. *The central transposition τ_c belongs to $S_{2^n}^*$.*

Proof. Let $\succ \in \mathcal{O}_n^*$ and let (x_k) be the order sequence corresponding to $\succ$. Then $\succ \in \overline{\mathcal{O}}_n$ and so $\overline{x}_{2^{n-1}} = x_{2^{n-1}+1}$. Thus, there does not exist a complete set $\langle a, b, c, d \rangle$ for which $a = 2^{n-1}$ and $b = 2^{n-1} + 1$ or for which $c = 2^{n-1}$ and $d = 2^{n-1} + 1$. Since $\langle 2^{n-1}, 2^{n-1} + 1 \rangle$ is the only pair inverted by τ_c, Proposition 44 implies that τ_c preserves the separability of $\succ$. Since our choice of $\succ$ was arbitrary, it follows that $\tau_c \in S_{2^n}^*$. $\square$

Corollary 48. *The reflection permutation τ_r belongs to $S_{2^n}^*$.*

Proof. For any complete set $\langle a, b, c, d \rangle$, τ_r inverts both $\langle a, b \rangle$ and $\langle c, d \rangle$. Thus, by Proposition 44, τ_r preserves the separability of any $\succ \in \mathcal{O}_n^*$, from which it follows that $\tau_r \in S_{2^n}^*$. $\square$

Corollary 49. *$S_{2^n}^*$ contains a subgroup isomorphic to V_4.*

Proof. Since $|\tau_c| = |\tau_r| = |\tau_c \tau_r| = 2$, the subgroup $S = \{1, \tau_c, \tau_r, \tau_c \tau_r\}$ contains exactly three non-identity elements, all of which have order 2. Thus, $S \cong V_4$. $\square$

The final step in establishing our main result is a string of lemmas leadingg to Proposition 62, a combinatorial result that will play a critical role in the proof of Theorem 29.

Let $\sigma \in S_{2^n}$ and let $\langle a, b \rangle$ be some pair. If both a and b are even, then we say that the pair $\langle a, b \rangle$ is *even*. Similarly, if both a and b are odd, then we say that $\langle a, b \rangle$ is *odd*. If $a, b \leqslant 2^{n-1}$, then we call $\langle a, b \rangle$ a *top-half pair*. Similarly, if $a, b > 2^{n-1}$, then we call $\langle a, b \rangle$ a *bottom-half pair*. The pair $\langle 2^{n-1}, 2^{n-1} + 1 \rangle$ is called the *central pair* and all other pairs are said to be *non-central*. We denote by $\mathrm{inv_e}(\sigma)$, $\mathrm{inv_o}(\sigma)$, $\mathrm{inv_t}(\sigma)$, and $\mathrm{inv_b}(\sigma)$ the respective numbers of even, odd, top-half, and bottom-half pairs inverted by σ. Notice that

$$0 \leqslant \mathrm{inv_e}(\sigma), \mathrm{inv_o}(\sigma), \ \mathrm{inv_t}(\sigma), \ \mathrm{inv_b}(\sigma) \leqslant \binom{2^{n-1}}{2}.$$

Lemma 50. *Let* $\sigma \in \overline{S}_{2^n}$. *Then* $\mathrm{inv_e}(\sigma) = \mathrm{inv_o}(\sigma)$ *and* $\mathrm{inv_t}(\sigma) = \mathrm{inv_b}(\sigma)$.

Proof. First observe that if $\langle a, b \rangle$ is an even pair, then $\langle \overline{b}, \overline{a} \rangle$ is an odd pair (and vice versa). Similarly, if $\langle a, b \rangle$ is a top-half pair, then $\langle \overline{b}, \overline{a} \rangle$ is a bottom-half pair (and vice versa). Now, let $\sigma \in \overline{S}_{2^n}$ and suppose that σ inverts $\langle a, b \rangle$. Then, by Lemma 40, σ also inverts $\langle \overline{b}, \overline{a} \rangle$. Consequently, the map $\langle a, b \rangle \mapsto \langle \overline{b}, \overline{a} \rangle$is a bijection between the set of even pairs inverted by σ and the set of odd pairs inverted by σ. It is also a bijection between the set of top-half pairs inverted by σ and the set of bottom-half pairs inverted by σ. $\square$

Lemma 51. *Let* $\sigma \in \overline{S}_{2^n}$ *and suppose that* $\mathrm{inv}(\sigma) \geqslant 2$. *Then either* $\mathrm{inv_e}(\sigma) > 0$ *or* $\mathrm{inv_t}(\sigma) > 0$.

Proof. Suppose, to the contrary, that both $\mathrm{inv_e}(\sigma) = 0$ and $\mathrm{inv_t}(\sigma) = 0$. Then for each pair $\langle a, b \rangle$ inverted by σ, it must be that $a \leqslant 2^{n-1}$, $b > 2^{n-1}$, and a and b have opposite parity. Since $\mathrm{inv}(\sigma) \geqslant 2$, there exists a non-central pair $\langle a, b \rangle$ inverted by σ. Since the pair is non-central and a and b have opposite parity, we have that $b - a > 2$ and so $a + 1 < b - 1$. Let $c = a + 1$ and $d = b - 1$. Then, $a < c < d < b$ and exactly one of the following conditions is satisfied:

(i) $a, c, d \leqslant 2^{n-1}$ and $b > 2^{n-1}$,
(ii) $a, c \leqslant 2^{n-1}$ and $d, b > 2^{n-1}$,
(iii) $a \leqslant 2^{n-1}$ and $c, d, b > 2^{n-1}$.

In case (i), we note that σ does not invert $\langle a, c \rangle$ since it is a top-half pair and $\mathrm{inv_t}(\sigma) = 0$. Furthermore, σ does not invert $\langle c, b \rangle = \langle a + 1, b \rangle$ since $a + 1$ and b have the same parity and $\mathrm{inv_e}(\sigma) = \mathrm{inv_o}(\sigma) = 0$. But σ does invert $\langle a, b \rangle$ and so we have that $\sigma(c) < \sigma(b) < \sigma(a) < \sigma(c)$, a contradiction. Similar contradictions can be reached for cases (ii) and (iii). $\square$

Lemma 52. *Let* $\sigma \in \overline{S}_{2^n}$, *where* $n \geqslant 5$, *and suppose that* $\mathrm{inv}(\sigma) \leqslant \frac{1}{2}\binom{2^n}{2}$. *Then either* $\mathrm{inv_e}(\sigma) < \binom{2^{n-1}}{2} - 1$ *or* $\mathrm{inv_t}(\sigma) < \binom{2^{n-1}}{2} - 1$.

Proof. We prove the contrapositive. Suppose that $\mathrm{inv_e}(\sigma), \mathrm{inv_t}(\sigma) \geqslant \binom{2^{n-1}}{2} - 1$. Let S_e, S_o, S_t, and S_b be the sets of even, odd, top-half, and bottom-half pairs inverted by σ. Then $|S_\mathrm{e}|, |S_\mathrm{t}| \geqslant \binom{2^{n-1}}{2} - 1$, and by Lemma 50, $|S_\mathrm{o}|, |S_\mathrm{b}| \geqslant \binom{2^{n-1}}{2} - 1$. Notice that $S_\mathrm{e} \cap S_\mathrm{o} = S_\mathrm{t} \cap S_\mathrm{b} = \emptyset$. Notice also that $|S_\mathrm{e} \cap S_\mathrm{t}| = |S_\mathrm{o} \cap S_\mathrm{b}| \leqslant \binom{2^{n-2}}{2}$ and that $|S_\mathrm{o} \cap S_\mathrm{t}| = |S_\mathrm{e} \cap S_\mathrm{b}| \leqslant \binom{2^{n-2}}{2}$.

Thus,

$$\begin{aligned}
\mathrm{inv}(\sigma) &\geqslant |S_\mathrm{e} \cup S_\mathrm{o} \cup S_\mathrm{t} \cup S_\mathrm{b}| \\
&= |S_\mathrm{e}| + |S_\mathrm{o}| + |S_\mathrm{t}| + |S_\mathrm{b}| - |S_\mathrm{e} \cap S_\mathrm{t}| - |S_\mathrm{o} \cap S_\mathrm{b}| - |S_\mathrm{o} \cap S_\mathrm{t}| - |S_\mathrm{e} \cap S_\mathrm{b}| \\
&\geqslant 4\left[\binom{2^{n-1}}{2} - 1\right] - 4\binom{2^{n-2}}{2} \\
&= 2(2^{n-1})(2^{n-1} - 1) - 2(2^{n-2})(2^{n-2} - 1) - 4 \\
&= 2^{2n-1} - 2^n - 2^{2n-3} + 2^{n-1} - 4 \\
&= 2^{2n-1} - 2^{n-1} - 2^{2n-3} - 4 \\
&= (2^{2n-2} - 2^{n-2}) + (2^{2n-2} - 2^{n-2} - 2^{2n-3} - 4)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\binom{2^n}{2} + 2^{2n-3} - (2^{n-2}+4) \\
&> \frac{1}{2}\binom{2^n}{2} + 2^{2n-3} - 2^{n-1} \quad (\text{since } n \geqslant 5) \\
&= \frac{1}{2}\binom{2^n}{2} + 2^{n-1}(2^{n-2}-1) \\
&> \frac{1}{2}\binom{2^n}{2}. \qquad \square
\end{aligned}
$$

Lemma 53. *Let $\sigma \in \overline{S}_{2^n}$, where $n \geqslant 4$.*

(i) *If* $\mathrm{inv_e}(\sigma) \geqslant \binom{2^{n-1}}{2} - 1$, *then* $\mathrm{inv_t}(\sigma) \geqslant 5$.
(ii) *If* $\mathrm{inv_t}(\sigma) \geqslant \binom{2^{n-1}}{2} - 1$, *then* $\mathrm{inv_e}(\sigma) \geqslant 5$.

Proof. We prove (i) and leave the analogous proof of (ii) to the reader. Suppose that $\mathrm{inv_e}(\sigma) \geqslant \binom{2^{n-1}}{2} - 1$. Then there is at most one even pair that is not inverted by σ. Since $\binom{2^{n-2}}{2}$ of the top-half pairs are also even, σ must invert at least $\binom{2^{n-2}}{2} - 1$ top-half pairs. But $n \geqslant 4$, and so it follows that

$$\mathrm{inv_t}(\sigma) \geqslant \binom{2^{n-2}}{2} - 1 \geqslant \binom{4}{2} - 1 = 5,$$

as desired. $\square$

Lemma 54. *For all odd integers a and b with $a < b \leqslant 2^n - 1$, $\langle a, b, a+1, b+1 \rangle$ is a complete set with respect to $\succ_{\mathrm{lex}}$.*

Proof. Let $\succ_{n-1}$ be the standard lexicographic order on $n-1$ questions and let $\succ_1$ be the preference order on one question specified by $1 \succ_1 0$. Then $\succ_{\mathrm{lex}} = \succ_{n-1} \oplus \succ_1$. Let (y_k), (x_k) be the order sequences corresponding to $\succ_{\mathrm{lex}}$ and $\succ_{n-1}$, respectively. Then, by Proposition 7, $y_{2k-1} = (x_k, 1)$ and $y_{2k} = (x_k, 0)$ for each $k \leqslant 2^{n-1}$. Now, let $a = 2i - 1$ and $b = 2j - 1$, where $i < j$. Then

$$
\begin{aligned}
y_a &= (x_i, 1), \quad & y_{a+1} &= (x_i, 0), \\
y_b &= (x_j, 1), \quad & y_{b+1} &= (x_j, 0),
\end{aligned}
$$

and so $\langle a, b, a+1, b+1 \rangle$ is a complete set. $\square$

Lemma 55. *Let $\succ_{\mathrm{lex}}$ be the standard lexicographic order on n questions. Then for all $a < b \leqslant 2^{n-1}$, $\langle a, b, a + 2^{n-1}, b + 2^{n-1} \rangle$ is a complete set with respect to $\succ_{\mathrm{lex}}$.*

Proof. The proof is analogous to that of Lemma 54. $\square$

Lemma 56. *Let $\sigma \in S^*_{2^n}$. If $\mathrm{inv_o}(\sigma) = 1$ and $\langle a, b \rangle$ is the unique odd pair inverted by σ, then $a + b = 2^n$.*

Proof. Suppose that $\mathrm{inv_o}(\sigma) = 1$ and let $\langle a, b \rangle$ be the unique odd pair inverted by σ. By Lemma 54, $\langle a, b, a+1, b+1 \rangle$ is a complete set with respect to $\succ_{\mathrm{lex}}$. But since σ preserves the separability of $\succ_{\mathrm{lex}}$, Proposition 44 implies that σ inverts $\langle a+1, b+1 \rangle$. By Lemma 40, σ also inverts $\langle \overline{b+1}, \overline{a+1} \rangle$, which is an odd pair since both $a+1$ and $b+1$

are even. But $\langle a, b \rangle$ is the unique odd pair inverted by σ and so it must be that $\langle \overline{b+1}, \overline{a+1} \rangle = \langle a, b \rangle$. Thus

$$a = \overline{b+1} = 2^n - (b+1) + 1 = 2^n - b,$$

as desired. □

Lemma 57. *Let $\sigma \in S^*_{2^n}$. If $\mathrm{inv}_t(\sigma) = 1$ and $\langle a, b \rangle$ is the unique top-half pair inverted by σ, then $a + b = 2^{n-1} + 1$.*

Proof. Suppose that $\mathrm{inv}_t(\sigma) = 1$ and let $\langle a, b \rangle$ be the unique top-half pair inverted by σ. By Lemma 55, $\langle a, b, a + 2^{n-1}, b + 2^{n-1} \rangle$ is a complete set with respect to $\succ_{\mathrm{lex}}$. Thus, σ must also invert $\langle a + 2^{n-1}, b + 2^{n-1} \rangle$ and $\langle \overline{b + 2^{n-1}}, \overline{a + 2^{n-1}} \rangle$. But $\langle \overline{b + 2^{n-1}}, \overline{a + 2^{n-1}} \rangle$ is a top-half pair and so it must be that

$$a = \overline{b + 2^{n-1}} = 2^n - (b + 2^{n-1}) + 1 = 2^{n-1} - b + 1. \quad \square$$

Lemma 58. *Let $\succ \in \mathcal{O}_n$ and let $\succ_1 \in \mathcal{O}_1$. If $\langle a, a+1, b, b+1 \rangle$ is a complete set with respect to $\succ$, then $\langle 2a, 2a + 1, 2b, 2b + 1 \rangle$ is a complete set with respect to $\succ \oplus \succ_1$.*

Proof. Without loss of generality, assume that $1 \succ_1 0$. Let (x_k), (y_k) be the order sequences corresponding to $\succ$ and $\succ \oplus \succ_1$, respectively. If $\langle a, a+1, b, b+1 \rangle$ is a complete set with respect to $\succ$, then there exists $S \subset Q$ and $y, z \in X_S$, $u, v \in X_{-S}$ such that

$$x_a = (y, u), \qquad x_b = (y, v),$$
$$x_{a+1} = (z, u), \quad x_{b+1} = (z, v).$$

Now by Proposition 7, we have

$$x_{2a} = (y, u, 0), \qquad x_{2b} = (y, v, 0)$$
$$x_{2a+1} = (z, u, 1), \quad x_{2b+1} = (z, v, 1).$$

But then $\langle 2a, 2a+1, 2b, 2b+1 \rangle$ is a complete set with respect to $\succ \oplus \succ_1$, as desired. □

Lemma 59. *Let $\sigma \in S^*_{2^n}$, where $n \geqslant 4$. If σ inverts $\langle 2^{n-2}, 2^{n-2} + 1 \rangle$, then there is a top-half pair $\langle a, b \rangle \neq \langle 2^{n-2}, 2^{n-2} + 1 \rangle$ such that σ inverts $\langle a, b \rangle$.*

Proof. It suffices to show that, for each $n \geqslant 4$, there exists an order $\succ \in \mathcal{O}^*_n$ such that $\langle 2^{n-2}, 2^{n-2} + 1, 2^{n-1} - 2^{n-4}, 2^{n-1} - 2^{n-4} + 1 \rangle$ is a complete set with respect to $\succ$. Indeed, once we have established this fact, our result follows directly from Proposition 44, since $\langle 2^{n-1} - 2^{n-4}, 2^{n-1} - 2^{n-4} + 1 \rangle$ is a top-half pair.

We proceed by induction. For $n = 4$, consider the separable order $\succ$ corresponding to the normalized binary preference matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Notice that $\langle 4, 5, 7, 8 \rangle$ is a complete set with respect to $\succ$, as desired. Now suppose that our claim is true for some $n \geqslant 4$ and let $\succ' \in \mathcal{O}_n^*$ be chosen so that $\langle 2^{n-2}, 2^{n-2}+1, 2^{n-1}-2^{n-4}, 2^{n-1}-2^{n-4}+1 \rangle$ is a complete set with respect to $\succ'$. Let $\succ_1$ be the preference order on one question specified by $1 \succ_1 0$. Then, by Lemma 58, $\langle 2^{n-1}, 2^{n-1}+1, 2^n-2^{n-3}, 2^n-2^{n-3}+1 \rangle$ is a complete set with respect to $\succ' \oplus \succ_1$. Proposition 6 implies that $\succ' \oplus \succ_1 \in \mathcal{O}_{n+1}^*$, which completes the proof. $\square$

Lemma 60. *Let $\sigma \in S_{2^n}^*$, where $n \geqslant 4$. Then* $\mathrm{inv_t}(\sigma) \neq 1$.

Proof. Suppose that $\mathrm{inv_t}(\sigma) = 1$ and let $\langle a, b \rangle$ be the unique top-half pair inverted by σ. Then, by Lemma 57, $a + b = 2^{n-1} + 1$. We consider two cases:

Case 1. $b = a + 1$. Then $a = 2^{n-2}$ and $b = 2^{n-2} + 1$. Consequently, by Lemma 59, there exists a top-half pair $\langle c, d \rangle \neq \langle a, b \rangle$ that is inverted by σ. This, however, is a contradiction to the assumption that $\mathrm{inv_t}(\sigma) = 1$.

Case 2. $b > a + 1$. Then $a < b - 1 < b$. Since σ inverts $\langle a, b \rangle$, Lemma 38 implies that σ inverts either $\langle a, b-1 \rangle$ or $\langle b-1, b \rangle$, both of which are top-half pairs. This too is a contradiction to the assumption that $\mathrm{inv_t}(\sigma) = 1$. $\square$

Lemma 61. *Let $\sigma \in S_{2^n}^*$, where $n \geqslant 4$. If* $\mathrm{inv_o}(\sigma) = 1$, *then* $\mathrm{inv_t}(\sigma) > 1$.

Proof. Suppose that $\mathrm{inv_o}(\sigma) = 1$ and let $\langle a, b \rangle$ be the unique odd pair inverted by σ. Then, by Lemma 56, $a + b = 2^n$. If $b \neq a + 2$, then there exists some odd integer c such that $a < c < b$. But then Lemma 38 implies that σ inverts either $\langle a, c \rangle$ or $\langle c, b \rangle$, a contradiction to the assumption that $\mathrm{inv_o}(\sigma) = 1$. Thus, it must be the case that $b = a + 2$, which implies that $a = 2^{n-1} - 1$ and $b = 2^{n-1} + 1$. Let $\succ_{\mathrm{lex}}$ be the standard lexicographic order on n questions and let (x_k), (y_k) be the order sequences corresponding to $\succ_{\mathrm{lex}}$ and $\tau_c(\succ_{\mathrm{lex}})$, respectively. Then, by Proposition 10,

$$\begin{aligned}
y_{2^{n-1}-1} &= x_{2^{n-1}-1} = (1, 0, 0, , \ldots, 0, 1),\\
y_{2^{n-1}+1} &= x_{2^{n-1}} = (1, 0, 0, , \ldots, 0, 0),\\
y_1 &= x_1 = (1, 1, 1, \ldots, 1, 1),\\
y_2 &= x_2 = (1, 1, 1, \ldots, 1, 0).
\end{aligned}$$

Thus, $\langle 1, 2, 2^{n-1}-1, 2^{n-1}+1 \rangle$ is a complete set with respect to $\tau_c(\succ_{\mathrm{lex}})$. Since $\sigma \in S_{2^n}^*$ and σ inverts $\langle a, b \rangle = \langle 2^{n-1}-1, 2^{n-1}+1 \rangle$, Proposition 44 implies that σ inverts $\langle 1, 2 \rangle$, which implies that $\mathrm{inv_t}(\sigma) \geqslant 1$. Lemma 60 then makes this inequality strict. $\square$

Proposition 62. *If $\sigma \in S_{2^n}^*$ and $2 \leqslant \mathrm{inv}\sigma \leqslant \frac{1}{2}\binom{2^n}{2}$, then either $1 < \mathrm{inv_o}(\sigma) < \binom{2^{n-1}}{2} - 1$ or $1 < \mathrm{inv_t}(\sigma) < \binom{2^{n-1}}{2} - 1$.*

Proof. Suppose, to the contrary, that neither $1 < \mathrm{inv_o}(\sigma) < \binom{2^{n-1}}{2} - 1$ nor $1 < \mathrm{inv_t}(\sigma) < \binom{2^{n-1}}{2} - 1$. Then one of the following must occur:

(i) $\mathrm{inv_o}(\sigma), \mathrm{inv_t}(\sigma) \leqslant 1$,

(ii) $\mathrm{inv_o}(\sigma) \leqslant 1$ and $\mathrm{inv_t}(\sigma) \geqslant \binom{2^{n-1}}{2} - 1$,

(iii) $\mathrm{inv_o}(\sigma) \geqslant \binom{2^{n-1}}{2} - 1$ and $\mathrm{inv_t}(\sigma) \leqslant 1$,

(iv) $\mathrm{inv_o}(\sigma), \mathrm{inv_t}(\sigma) \geqslant \binom{2^{n-1}}{2} - 1$.

Lemmas 52 and 53 rule out cases (ii)–(iv). Thus, it must be the case that both $\mathrm{inv_o}(\sigma), \mathrm{inv_t}(\sigma) \leqslant 1$. Lemma 60 implies that $\mathrm{inv_t}(\sigma) = 0$. But then Lemma 51 implies that $\mathrm{inv_o}(\sigma) = 1$. This, however, is a contradiction, since, by Lemma 61, $\mathrm{inv}_{\mathrm{o}}(\sigma)=1 \quad\Rightarrow\quad \mathrm{inv}_{\mathrm{t}}(\sigma)>1$. Thus, it must be the case that either $1<\mathrm{inv}_{\mathrm{o}}(\sigma)<\binom{2^{n-1}}{2}-1$ or $1<\mathrm{inv}_{\mathrm{t}}(\sigma)<\binom{2^{n-1}}{2}-1$, as desired. $\square$

We are now able to prove Theorem 29.

Proof of Theorem 29. For $n=2$, separability and symmetry are equivalent. Thus, $S_4^*=\overline{S}_4$, which is, by our remarks following Proposition 18, isomorphic to D_4, the group of symmetries of the square.

For $n=3$ and 4, a brute-force method can be used to calculate $S_{2^n}^*$. Note that, by Corollary 46, a permutation σ belongs to $S_{2^n}^*$ if and only if σ preserves the separability of each normalized preference order in $\mathcal{O}_n^*$, that is, if and only if $\sigma\in S_{2^n}^{\succ}$ for each normalized $\succ\,\in\mathcal{O}_n^*$. Section 2.2 provides explicit descriptions of all such orders. Furthermore, by taking permutations and/or bitwise complements of the columns of these normalized preference orders, we can explicitly determine *all* of the elements of $\mathcal{O}_n^*$. We can then calculate $S_{2^n}^{\succ}$ for each normalized $\succ\,\in\mathcal{O}_n^*$, using Proposition 27 to note that

$$S_{2^n}^{\succ}=\bigcap_{\succ'\in\mathcal{O}_n^*}\{\sigma\in S_{2^n}\colon \sigma(\succ)=\succ'\}.$$

Finally, we obtain $S_{2^n}^*$ by taking the intersection of these $S_{2^n}^{\succ}$.

The case for $n=3$ can be done by hand (albeit somewhat tediously), since only two sets of permutations must be intersected. The result is a group of 24 permutations, generated by $\sigma=(18)(27)(36)(45)$, $\tau=(18)(26)(37)$, and $\gamma=(12)(36)(45)(78)$. Notice that $\sigma\tau=(23)(45)(67)$, $\sigma\gamma=(17)(28)$, and $\tau\gamma=(167)(283)(45)$. Thus,

$$\langle\,\sigma,\tau,\gamma\mid\sigma^2=\tau^2=\gamma^2=(\sigma\tau)^2=(\sigma\gamma)^2=(\tau\gamma)^6=1\,\rangle$$

is a presentation for S_8^*. By Coxeter and Moser [4], it follows that $S_8^*\cong Z_2\times D_6\cong V_4\times S_3$.

The situation is slightly more complicated for $n=4$, since we are now intersecting 14 sets, each containing 5376 permutations. Using a computer program to complete this task, we find that $S_{16}^*=\{1,\tau_{\mathrm{c}},\tau_{\mathrm{r}},\tau_{\mathrm{c}}\tau_{\mathrm{r}}\}\cong V_4$ (see [5] for computer code and further details). Indeed, our claim is that $S_{2^n}^*=\{1,\tau_{\mathrm{c}},\tau_{\mathrm{r}},\tau_{\mathrm{c}}\tau_{\mathrm{r}}\}\cong V_4$ for all $n\geqslant 4$.

Proceeding by induction, suppose that, for some $n\geqslant 5$, $S_{2^{n-1}}^*=\{1,\tau_{\mathrm{c}},\tau_{\mathrm{r}},\tau_{\mathrm{c}}\tau_{\mathrm{r}}\}$. Suppose also that $S_{2^n}^*\neq\{1,\tau_{\mathrm{c}},\tau_{\mathrm{r}},\tau_{\mathrm{c}}\tau_{\mathrm{r}}\}$. Corollary 49 implies that $\{1,\tau_{\mathrm{c}},\tau_{\mathrm{r}},\tau_{\mathrm{c}}\tau_{\mathrm{r}}\}\subseteq S_{2^n}^*$, and so it must be the case that this containment is proper. Thus, by our comments following Definition 31 and by Proposition 41 and Lemma 42, there exists $\sigma\in S_{2^n}^*$ with $2\leqslant\mathrm{inv}\,\sigma\leqslant\binom{2^n}{2}-2$. Furthermore, we may assume, by Lemma 34, that $2\leqslant\mathrm{inv}\,\sigma\leqslant\frac{1}{2}\binom{2^n}{2}$. But then Proposition 62 implies that either $1<\mathrm{inv}_{\mathrm{o}}(\sigma)<\binom{2^{n-1}}{2}-1$ or $1<\mathrm{inv}_{\mathrm{t}}(\sigma)<\binom{2^{n-1}}{2}-1$.

For the former, suppose that $1<\mathrm{inv}_{\mathrm{o}}(\sigma)<\binom{2^{n-1}}{2}-1$ and let $\succ_1$ be the preference order on one question specified by $1\succ_1 0$. Then for any $\succ\,\in\mathcal{O}_{n-1}^*$, the permutation $\sigma'=s_n(\sigma)$ induced by $p_n(\succ\,\oplus\succ_1)$ is an element of $S_{2^{n-1}}^*$ (by Proposition 37). On the other hand, Propositions 10 and 32 establish a bijection between the odd pairs inverted by σ and all pairs inverted by σ', and so $\mathrm{inv}(\sigma')=\mathrm{inv}_{\mathrm{o}}(\sigma)$. This, however, is a contradiction, since $1<\mathrm{inv}_{\mathrm{o}}(\sigma)<\binom{2^{n-1}}{2}-1$ and $\sigma'\in S_{2^{n-1}}^*$, which is equal to $\{1,\tau_{\mathrm{c}},\tau_{\mathrm{r}},\tau_{\mathrm{c}}\tau_{\mathrm{r}}\}$ by the induction hypothesis.

A similar contradiction is obtained if we assume that $1<\mathrm{inv}_{\mathrm{t}}(\sigma)<\binom{2^{n-1}}{2}-1$ (in this case we consider the permutation $\sigma'=s_1(\sigma)$ induced by $p_1(\succ_1\oplus\succ)$). Since each case leads to contradiction, our assumption that $S_{2^n}^*\neq\{1,\tau_{\mathrm{c}},\tau_{\mathrm{r}},\tau_{\mathrm{c}}\tau_{\mathrm{r}}\}$ must be false. $\square$

The conclusion of Theorem 29 (namely that $S_{2^n}^*$ contains only four elements for $n\geqslant 4$) is the last in a series of disturbing results about separable preferences. In addition to past research documenting the paradoxical behavior that can occur in the presence of nonseparable preferences, we showed via Corollary 15 that the desirable property of separability is exceedingly rare. Through Theorem 29, we established further that most changes to separable preference orders (even small modifications such as the interchanging of two non-central, adjacent outcomes) have the potential to introduce nonseparability.

The algebraic structure of $S^*_{2^n}$ (namely, that this set of separability-preserving permutations is a group) is less significant than its small size. However, the lack of algebraic structure exhibited by other related sets of permutations is significant. We mentioned earlier that for $n \geqslant 3$, there does not exist a separable order $\succ \in \mathcal{O}^*_n$ for which $S^{\succ}_{2^n}$ is a subgroup of $\overline{S}_{2^n}$. We now prove this result and discuss its importance.

Lemma 63. *For $n \geqslant 3$, $S^{\text{lex}}_{2^n}$ is not a subgroup of $\overline{S}_{2^n}$.*

Proof. Let $n \geqslant 3$ be given and let (y_k) be the order sequence corresponding to $\succ_{\text{lex}}$. Let

$$\sigma = (2, 2 + 2^{n-1} - 1)(4, 4 + 2^{n-1} - 1) \cdots (2^{n-1}, 2^{n-1} + 2^{n-1} - 1).$$

Proposition 10 implies that, for every $x \in X_{Q-\{1,n\}}$, $y_m = (1, x, 0)$ if and only if $m = 2k$ for some $k \leqslant 2^{n-2}$. Now let $y_{2k} = (1, x, 0)$, as above. We claim that $y_{2k+2^{n-1}-1} = (0, x, 1)$. To prove this, it suffices to show that

$$2^n - (2k + 2^{n-1} - 1) = 0 \cdot 2^{n-1} + \sum_{j=2}^{n-1} x_j \cdot 2^{n-j} + 1 \cdot 2^0 = \sum_{j=2}^{n-1} x_j \cdot 2^{n-j} + 1.$$

Now we know that

$$2^n - 2k = 2^{n-1} + \sum_{j=2}^{n-1} x_j \cdot 2^{n-j},$$

and so

$$\sum_{j=2}^{n-1} x_j \cdot 2^{n-j} = 2^n - 2k - 2^{n-1}.$$

Thus

$$\sum_{j=2}^{n-1} x_j \cdot 2^{n-j} + 1 = 2^n - 2k - 2^{n-1} + 1 = 2^n - (2k + 2^{n-1} - 1),$$

as desired.

It follows from our above observations that σ interchanges in (y_k) all pairs of elements of X_Q of the form $((1, x, 0), (0, x, 1))$ for $x \in X_{Q-\{1,n\}}$. The effect of this action is to exchange the first and the nth questions in each outcome while maintaining the original order specified by $\succ_{\text{lex}}$. This exchange has no effect on the separability of the order and so $\sigma(\succ_{\text{lex}})$ is separable, which implies that $\sigma \in S^{\text{lex}}_{2^n}$.

Now let $\gamma = \tau_c \sigma$. Then $\gamma(\succ_{\text{lex}}) = \tau_c(\sigma(\succ_{\text{lex}}))$ is separable since $\tau_c \in S^*_{2^n}$. Thus, $\gamma \in S^{\text{lex}}_{2^n}$. Notice also that

$$\begin{aligned}\gamma &= (2^{n-1}, 2^{n-1} + 1)(2, 2 + 2^{n-1} - 1)(4, 4 + 2^{n-1} - 1) \cdots (2^{n-1}, 2^{n-1} + 2^{n-1} - 1) \\ &= (2, 2^{n-1}, 2^n - 1, 2^{n-1} + 1)(4, 4 + 2^{n-1} - 1) \cdots (2^{n-1} - 2, 2^{n-1} - 2 + 1)\end{aligned}$$

and so $\gamma^2 = (2^{n-1}, 2^{n-1} + 1)(2, 2^n - 1) = \tau_c(2, 2^n - 1)$. We claim that $\gamma^2 \notin S^{\text{lex}}_{2^n}$. Since $\tau_c \in S^*_{2^n}$, it suffices to show that $(2, 2^n - 1) \notin S^{\text{lex}}_{2^n}$. Let (z_k) be the order sequence corresponding to $\succ = (2, 2^n - 1)(\succ_{\text{lex}})$. Then

$$\begin{aligned}z_1 &= y_1 = (1, 1, \ldots, 1, 1), \\ z_2 &= y_{2^n - 1} = (0, 0, \ldots, 0, 1), \\ z_{2^{n-1}-1} &= y_{2^{n-1}-1} = (1, 0, \ldots, 0, 1), \\ z_{2^{n-1}+1} &= y_{2^{n-1}+1} = (0, 1, \ldots, 1, 1).\end{aligned}$$

Thus, $(\mathbf{1}, 1, \ldots, 1, 1) \succ (\mathbf{0}, 1, \ldots, 1, 1)$ but $(\mathbf{0}, 0, \ldots, 0, 1) \succ (\mathbf{1}, 0, \ldots, 0, 1)$, and so $\succ = (2, 2^n - 1)(\succ_{\text{lex}})$ is not separable. It follows that $\gamma^2 \notin S_{2^n}^{\text{lex}}$ and so $S_{2^n}^{\text{lex}}$ is not a subgroup of $\overline{S}_{2^n}$. $\square$

Lemma 64. *Let $\succ_1, \succ_2 \in \mathcal{O}_n^*$. If $S_{2^n}^{\succ_1}$ is a subgroup of $\overline{S}_{2^n}$, then so is $S_{2^n}^{\succ_2}$.*

Proof. Suppose that $S_{2^n}^{\succ_1}$ is a subgroup of $\overline{S}_{2^n}$. We wish to show that $S_{2^n}^{\succ_2}$ is a also subgroup of $\overline{S}_{2^n}$. Since $S_{2^n}^{\succ_2}$ is finite and $1 \in S_{2^n}^{\succ_2}$, it suffices to show that $S_{2^n}^{\succ_2}$ is closed under its operation. To this end, choose $\tau_1, \tau_2 \in S_{2^n}^{\succ_2}$ and let $\sigma \in S_{2^n}$ be such that $\sigma(\succ_1) = \succ_2$. Since $\sigma(\succ_1) = \succ_2 \in \mathcal{O}_n^*$, it follows that $\sigma \in S_{2^n}^{\succ_1}$. Thus, $\sigma^{-1} \in S_{2^n}^{\succ_1}$. Now

$$\tau_1\sigma(\succ_1) = \tau_1(\sigma(\succ_1)) = \tau_1(\succ_2) \in \mathcal{O}_n^*$$

since $\tau_1 \in S_{2^n}^{\succ_2}$, from which it follows that $\tau_1\sigma \in S_{2^n}^{\succ_1}$. Similarly, $\tau_2\sigma \in S_{2^n}^{\succ_1}$. But then

$$\tau_1\tau_2\sigma = (\tau_1\sigma)(\sigma^{-1})(\tau_2\sigma) \in S_{2^n}^{\succ_1},$$

which implies that $\tau_1\tau_2(\succ_2) = \tau_1\tau_2\sigma(\succ_1) \in \mathcal{O}_n^*$. It follows that $\tau_1\tau_2 \in S_{2^n}^{\succ_2}$, as desired. $\square$

Proposition 65. *Let $n \geqslant 3$ and let $\succ \in \mathcal{O}_n^*$. Then $S_{2^n}^{\succ}$ is not a subgroup of $\overline{S}_{2^n}$.*

Proof. If $S_{2^n}^{\succ}$ were a subgroup of $\overline{S}_{2^n}$, then, by Lemma 64, $S_{2^n}^{\text{lex}}$ would also be a subgroup of $\overline{S}_{2^n}$. This, however, is a contradiction to Lemma 63. $\square$

We note here that, for $n = 2$, $S_{2^n}^{\succ}$ is always a subgroup of $\overline{S}_{2^n}$. In fact, for every $\succ \in \mathcal{O}_2^*$, $S_4^{\succ} = S_4^*$. To see this, notice that $S_4^* \subseteq S_4^{\succ}$ for every $\succ \in \mathcal{O}_2^*$. Also notice that $|S_4^{\succ}| = |\mathcal{O}_2^*| = 8 = |S_4^*|$ by Theorem 29 and by Propositions 9, 16, and 27. Thus, $S_4^{\succ} = S_4^*$, as desired.

Proposition 65 suggests a level of complexity within separable preferences that in some sense undermines any potential efforts to eliminate or control the presence of preference interdependencies within referendum elections. While Theorem 29 demonstrates that only a few permutations preserve the separability of *all* preference orders, one might still hope to identify larger sets of permutations that *locally* preserve separability (that is, preserve the separability of a particular preference order). Proposition 65 implies that while we may be able to find such sets, they will not necessarily be well-behaved in the sense of possessing a group structure. Specifically, they will not be closed under compositions, which means that once we have applied a particular permutation to a preference order, we will have to start from scratch if we wish to apply further permutations and still maintain the property of separability.

Of course, most voters in actual elections (a few mathematicians excluded) do not consciously view their preferences in this light. But voter preferences do evolve over time. What the average voter may not realize, but what our results suggest, is that the steps in this evolutionary process have the potential to introduce complexities that can ultimately affect not only the voter's ability to voice his or her preferences, but also the election outcomes that may occur as a result of these complex preferences.

6. Summary and conclusions

The property of separability is desirable for effective group decision-making, but our results suggest that separability is also rare and rather ill-behaved with regard to changes in voter preferences. These facts cast doubt on the feasibility of recent proposals to solve the separability problem by attempting to simply avoid interdependent preferences. Consequently, further research into other potential solutions, such as election sequencing and alternative aggregation methods, may be warranted.

From a theoretical perspective, many questions remain in the study of separable preference orders. These theoretical questions are interesting in their own right, but may also one day lead to more effective ways of dealing with the separability problem. One possible direction for future work would be to attempt to exploit Proposition 27 in order to derive a formula for the number of separable preference orders on a given question set. On a related note, Kilgour [6] seems to have been the first to observe the existence of a one-to-one correspondence between the collection of all monoseparable orders on a set Q (that is, the set of all orders that are separable on all individual questions but not necessarily on larger groups of questions) and the set of all linear extensions of the Boolean algebra on Q. This correspondence suggests that solutions to combinatorial problems involving separable preference orders have the potential to shed light on other important questions from the broader field of applied combinatorics.

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